

GAMES OF STATUS^{1,2}

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In a study intended to point toward possible applications of game-theoretic reasoning to sociological problems, mathematical models are used initially to examine the results of two-person nonconstant sum games in which a single value, status, is maximized, and subsequently to examine the results of multi-person games. The two-person, single-value-maximizing game quickly demonstrates behavior to be purely competitive. The multi-person game illustrates possibilities of coalition behavior, as demonstrated elsewhere in game theory, which includes the possibility of combinations against individual players, but the outcomes are highly sensitive to the values assigned to status factors, as in peck orders and potlatches.

The relationship between games of status and simple games is examined also.

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INTRODUCTION

STATUS or position is often more important in a society or in socio-political and socio-economic situations than wealth or other physical goods.

In an affluent society many individuals can be rich. However, (barring ties) there can only be one richest man. Some individuals strive to be good at that which they do. Others strive to be better than their peers, and some are driven to try to be best.

The potlatch and the pecking order, obtaining more power over people, keeping up with the Joneses or exceeding them, saving face or gaining face, are all manifestations of human or animal social behavior.

In experimental gaming, in many parlor games, examinations, competitions, and in many social situations where the payoffs are apparently relatively well defined and measurable, the behavior of individuals does not always support the hypothesis that the point score of the game, the grade on the examination, or the performance criterion in the competition is being maximized. Nor do the behavior patterns even support the hypothesis that a utility function positively correlated with the score is maximized.

Experimental work on two-person non-zero-sum games has shown that in some instances individuals play to maximize the difference between their scores. Thus, if we denote the strategy space of the i th player by S_i and the payoff (measured in whatever points, money, grades, or numbers are used in the game) by $P_i(s_1, s_2, s_3, \dots, s_n)$, then in the two-person game the players act as though they are playing a strictly competitive game with a payoff of:

$$P_1(s_1, s_2) - P_2(s_1, s_2)$$

to the first player and the negative of this to the second. In this instance the non-zero-sum game has been converted into a straightforward zero-sum game. When status is at stake, this is further reduced to a game with only three outcomes or, at most, only three important sets of outcomes. They are, respectively, win, draw, or lose, where win is defined as doing better than the competitor in the sense of making more points.

Two variants of this type of game in the two-person case merit distinction. The first can be called a game of pure status and the second a game of status and welfare. In the first game the point score obtained by both

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² The author is, as usual, indebted to L S Shapley for his comments and criticisms. Dr. Shapley's ability to find errors in my calculations

is equalled only by his skill in literary research as is illustrated by the quotation sent previously to this author when we were working on a joint paper: "My dear Shubrick [sic] if you keep as bad a reckoning at sea as you do ashore, God help the idlers who will all be drowned some night in their hammocks."

players is only of interest to them in determining who has been able to win or do better, who has emerged as first. In the second type of game, status is the most important aspect of the individual's utility; however, as long as further actions will not influence his status, he has a positive value on increasing joint welfare in terms of the distribution of wealth or services obtainable from the game. A player's valuation may be lexicographic in two dimensions in a game of status and welfare. The philosopher king, dictator, politician, corporate president, or dean may easily have the welfare of all others as his secondary concern, provided that this welfare does not challenge his position as first, in whatever system of precedence he calculates his rank. As is well known, for example, the ranking of an individual's position in a society has much to do with any standard of living calculation. Hence, apart from the difficulties encountered in defining relative price levels and representative bundles of goods to make up standard of living indices, any attempt at comparisons to determine whether or not an individual at a certain position in one society has a higher standard of living or lives better is beset with extra problems concerning society.

The type of situation envisioned here is particularly relevant to problems of rehabilitation, integration, and acceptance. It may not be enough to provide improvements in physical conditions; improvements in status must also be forthcoming.

Mathematical model construction is, at best, a dangerous art. This paper tries at most to highlight some of the aspects of status and to define a class of games that is both amenable to mathematical analysis and of sociological interest. Eventually, it would be desirable to develop an adequate theory of games of status and welfare; however, the mathematical difficulties are such that we first concentrate on games of status alone.

TWO-PERSON GAMES

A series of examples are presented to help illustrate some of the above remarks.

		Player 2's Strategies			
		1	2		
Player 1's Strategies	1	130,	130	10,	205
	2	200,	10	14,	14

EXAMPLE 1

In each cell shown above the payoff to the first player appears first and is followed by the payoff to the second player. The equilibrium point in this game is (2, 2) where each player obtains 14. Transforming this into the game $P_1(s_1, s_2) - P_2(s_1, s_2)$, we obtain

		1	2
1	0	-195	
2	190	0	

EXAMPLE 2

Suppose a loss is valued at -1, a draw as 0, and a win as X. The matrix for the game of pure status becomes:

		1	2
1	0	-1	
2	X	0	

EXAMPLE 3

In all of these examples, the solution to the game is given by the strategy pair (2, 2).

The second game, when viewed as an ordinary noncooperative game, is slightly different from the first:

		1	2	
1	130,	131	10,	205
2	200,	10	15,	14

EXAMPLE 4

However, it transforms to a difference game of:

		1	2
1	-1	-195	
2	190	+1	

EXAMPLE 5

or a status game with a payoff of

		1	2
1	-1	-1	
2	X	X	

EXAMPLE 6

which is always won by the first player.

The third game

	1	2
1	131, 130	10, 205
2	200, 10	14, 15

EXAMPLE 7

transforms to:

	1	2
1	X	-1
2	X	-1

EXAMPLE 8

which is always won by the second player.

If the game is played purely for status, then every two-person game becomes strictly competitive regardless of whether it is non-zero-sum or whether utilities are comparable. Furthermore, it makes no difference whether side-payments of points can be made, although this is not true in games with $n > 2$.

If both status and welfare are involved in the utilities, with status lexicographically more important, then the games in all instances above split into a strictly competitive and a cooperative stage. The first stage determines who wins and the second then maximizes and divides joint welfare. For example, in Example 7 the second player will always win the game; however, it is to their mutual benefit to divide 261 points between them rather than 210.

THREATS, C-GAMES, AND THE CHARACTERISTIC FUNCTION

A useful shorthand way of describing what is at stake in a multiperson decision is the characteristic function of an n -person game. This function which has 2^n values describes what any coalition can achieve by itself. Thus, eight values are required for the full description of the coalition possibilities in a three-person game. An example is given below:

$$\begin{aligned}
 v(\emptyset) &= 0 \\
 v(1) &= a_1 & v(2) &= b_1 & v(3) &= c_1 \\
 v(1, 2) &= a_2 & v(2, 3) &= b_2 & v(1, 3) &= c_2 \\
 v(1, 2, 3) &= 1.
 \end{aligned}$$

The assignment of zero to the coalition of no one is merely for formal neatness and we

need not note it further. The values given to all one-person coalitions may be regarded as specifying the initial peck order in the society. One normalization can be made. The unit of measurement of gain has one degree of freedom; i.e., we can multiply all outcomes by a scale factor k without making any difference to any strategic aspect of the game (it is as though we decided to change currency by issuing one new dollar for every ten old dollars). We may use this freedom to fix the value of $v(N) = 1$ where N is the set of all players.

It is usually assumed that a characteristic function is superadditive, i.e.,

$$v(S \cup T) \geq v(S) + v(T)$$

where S and T do not overlap,

or, in words, this states that a coalition of the members of group S and group T can, by acting jointly, obtain as much or more than they would separately.

Many economic, political, and social situations reflect this superadditive property. Cooperation usually pays. "We must all hang together, else we shall all hang separately." In economic situations, the numbers in the characteristic function might be dollars to be gained by cooperating; in parlor games, they may be points; in other circumstances, they might be units which are extremely difficult to measure.

The characteristic function represents an attempt to present a great amount of information in an extremely compact manner. It is a statistic or a coding or an abstraction. As such, it might conceal details that are of importance. Depending upon the specific situation and upon the question to be answered, the characteristic function may be most adequate, or it may be a totally distorted representation of the relevant facts.

Von Neumann and Morgenstern (1944) suggested that the characteristic function be calculated by assuming that while the group of players S is trying to maximize its gain, the remaining players $N - S$ are trying to minimize the gain to S . It is as though there were a two-group game being played with payoffs $P_S(A_S, A_{N-S})$ and $P_{N-S}(A_S, A_{N-S})$ where A_S and A_{N-S} are the strategies used by groups S and $N - S$, and P_S and

P_{N-S} are their respective payoffs. Thus

$$v(S) = \max\text{-min } P_S(A_S, A_{N-S}).$$

This idea is certainly reasonable for all constant sum games and some nonconstant sum games, but not for others. A constant sum game is one in which the amount that any coalition plus the amount that the opposing coalition obtains equals a constant. When a (0, 1) (Von Neumann & Morgenstern, 1944) normalization for the characteristic function is used this states that:

$$v(S) + v(N - S) = 1.$$

If a game is constant sum, it is then easy to see that the max-min method for evaluating the amount obtained by a coalition is reasonable. The assumption that $N - S$ tries to minimize the amount obtained by S is equivalent to stating that $N - S$ tries to maximize the amount it obtains. In other words, its welfare is directly in opposition to the welfare of S .

Most problems, be they social, economic, or political, are not best modeled by constant sum games. The amount lost by one side is often by no means equal to the amount obtained by the other. When this is the case, the max-min way for evaluating the worth of a coalition may grossly overstate or distort the description of the threat structure of the game. Several examples illustrate this.

Consider the following games; first

		Player 2	
		1	2
Player 1	1	5, 5	0, 10
	2	-1000, 0	-1000, 0

In this game if we use the von Neumann and Morgenstern max-min, we obtain (the non-normalized characteristic function):

$v(1) = 0$ (obtained by playing his first strategy),

$v(2) = 0$ (obtained by playing any strategy),

and

$v(1, 2) = 10$ when they cooperate.

But this does not reflect the actual threat

structure of this game. The characteristic function is symmetric, but the game is not. Suppose the status quo had been the strategy pair (1, 2) with the outcome (0, 10), i.e., with all of the gain going to the second individual. The first individual could threaten to use his second strategy in order to bargain for part of the 10, but he would run the risk of losing 1,000; hence, unless we were to regard him as aggrieved, paranoid, or concerned with a payoff considerably different from one described, his threat is not very realistic and the game actually favors the second player.

A different way to calculate a characteristic function that corrects for the asymmetry in threats is to consider max-min ($P_S - P_{N-S}$) where P_S and P_{N-S} are as previously defined. The displayed matrix represents the so-called threat game. Its max-min yields -10

		1	2
1		0	-10
2		-1000	-1000

as the difference $h(1) - h(2)$. The sum $h(1) + h(2)$ is given by max-max ($P_1 + P_2$) = 10. Thus we have the h -function

$$h(1) = 0 \quad h(2) = 10 \quad h(1, 2) = 10,$$

which stresses the advantage of the second player. This type of modification to the characteristic function was first suggested by Harsanyi (1959).

In the second game, cooperation is very naturally called for:

		1	2	3
1		10, 12	0, 0	0, 0
2		0, 0	0, -4	-10, -14
3		-30, -40	-50, -60	-60, -64

Here we have

$$v(1) = 0 \quad v(2) = -40 \quad v(1, 2) = 22$$

and a difference game of

-2	0	0
0	4	4
10	10	4

giving

$$h(1) = 13 \quad h(2) = 9 \quad h(1, 2) = 22.$$

One way of looking at the difference between the two characteristic functions presented for the last game is that the first accepts the possibility of extremely costly threat behavior completely leaving out considerations of the damage inflicted by the threatener upon himself, while the second evaluates threats in terms of a damage exchange rate between the players. In some situations the first is a reasonable representation of the problems, and in other situations, the second is. It is not possible to select between them purely on a priori grounds. The blind application of methodology is not sufficient to produce a relevant model of the social process. In particular, in many economic processes, the damage exchange rate calculation is likely to be a reasonable way to evaluate threats. In social processes it is possible that when an alienated group says they are going to do great damage even at the risk of their own destruction, they may mean it—in which case, the max-min calculation provides a better model of reality.

Games that have the property where, for the purpose at hand, the characteristic function provides an adequate representation of the strategic and combinatoric structure are called *c*-games (Shapley & Shubik, unpublished). Many of the game theoretic models of economic markets have this property. In political and diplomatic affairs the threat structure is critical and the resultant game models are often not *c*-games.

Although in this paper threat strategies are not going to be investigated further, as use is made of the characteristic function and related ways of describing the amounts obtained by coalitions, it is important to be aware of the difficulties in construction and the limitations in the use of the characteristic function.

GAMES OF STATUS

Description of a game of status

Suppose that a group of individuals are playing a game for points of some type. Furthermore, suppose that competing coalitions attempt to maximize the difference in their score. This is a straightforward generalization of the concept of damage exchange rate between two individuals to groups.

Suppose that coalitions S and $N - S$ have formed; they obtain $v(S)$ and $v(N - S)$. Now they play a new game in which they each distribute the points among members of their coalitions. The true payoff to the game is not the amount of points an individual obtains, but his status, which is determined by the rank order of the amount he obtains. Each individual's preferences are defined not on the points obtained, but on the n ranks from first to n th.

In order to complete the description of a game of status, a rule for handling ties must be given and the utility functions of the players described. Any rule will be somewhat arbitrary; a middling and an extreme convention are examined. When individuals in any group tie in the number of points obtained, we may either assign them the average of the available order, or assign all the bottom. For example, if three are tied for first, under the first convention they are all second, and under the second convention they are all third.

It is reasonable to consider probability mixes among the outcomes. Thus, an individual should be able to state his preferences between a choice of say, a 50-50 chance of being first or third, and a choice of being second with certainty. Any individual, i , will have a utility function $U_i(k)$ which describes the value he attaches to obtaining the k th position.

Obviously, in many instances the values attached to the position occupied will not necessarily vary in direct inverse proportion to the position. Thus, a man might evaluate being first as worth 10,000; second, 1,000, and third, 50. However, purely for ease of description, let us suppose that each individual attaches a value of n to being first, and a value of $n - k + 1$ to being k th.

Each coalition simultaneously and independently distributes the points it has available to its members. After it has done so, by looking at the points obtained by every member in the society,³ a rank can be as-

³ Implicitly, it is assumed that point scores are not allowed to be negative. It would be interesting to investigate the implications of relaxing this condition.

signed to each and the final payoffs are determined.

Although points in the apparent game are transferable, the actual game is one without side payments. An example of a four-person game of status will help to clarify this. Consider a symmetric four-person game with the following characteristic function:

$$f(1) = 0$$

$$f(2) = 4$$

$$f(3) = 7$$

$$f(4) = 10$$

(where $f(s)$ stands for the value of a coalition S with any s players as members).

Because there is not a single outcome to a coalition in a game of status, but a set of outcomes representing the value of the positions obtained by each member, we use a characterizing function $F(S)$ which describes the set of values obtainable by a coalition S .

If we assume that a tie obtains the lowest rank, then the characterizing function for the four-person game is as follows.⁴

$$F(1) = \{1\}$$

$$F(2) = \{2, 4\}, \{4, 2\}$$

$$F(3) = \{4, 3, 2\}, \{4, 2, 3\}, \{3, 2, 4\}, \\ \{3, 4, 2\}, \{2, 4, 3\}, \{2, 3, 4\}$$

$$F(4) = \{4, 3, 2, 1\} \text{ and all other permutations.}$$

Leaving aside mixed strategies, the best that a coalition of two can guarantee itself is $\{2, 4\}$ or $\{4, 2\}$, which can be obtained by distributing their points as $(4, 0)$ or $(0, 4)$. The best attack against this strategy is to reply with $(4, 0)$ or $(0, 4)$, resulting in a tie for first and third places.

If mixed strategies are allowed, then the coalition of two can guarantee itself an expected rank of $2\frac{1}{2}$ by using a continuous mixed strategy. The strategic problem of distributing points among the members of a

coalition can be seen to be a generalized Blotto game (Dresher, 1961).

In the four-person game, any three-person coalition can guarantee the first three positions.

The directed threat: α and β theories

An important feature of the struggle for status is that sometimes unsatisfied groups may select their target. Thus, a group may decide to concentrate all of its effort on getting a specific individual.

Two variants of the above game show a difference in the ability to attack a single individual. Suppose that the coalition S is required to specify a (pure) strategy stating how the points it has are to be divided among its members. This is done before $N - S$ has chosen its strategy, and the opposing coalition is given this information prior to making its choice. In this case, the opposing coalition can gang up against any individual in S and no one can help him. When this is contrasted with the game in which both sides move simultaneously, a difference can be seen. When the attacking coalition moves last, any coalition of two can guarantee at best the outcomes $\{2, 4\}$ or $\{4, 2\}$ and the opposing coalition can insure that it obtains no more. This is not the case when they move simultaneously. There is a difference between that which a coalition S can guarantee for itself and that which the coalition $N - S$ can prevent it from obtaining. By playing $(4, 0)$, the coalition of two cannot guarantee more than $\{2, 4\}$ for itself, as the opponents might also play $(4, 0)$; on the other hand, the opponents are not able to guarantee that they can hold the coalition to as low as $\{2, 4\}$. It might happen that they obtain $\{2, 3\}$ by using, for example, $(3, 1)$ against $(4, 0)$.

In no-side-payment games the distinction between: (α) what a coalition can guarantee itself, and (β), how much another coalition can prevent it from obtaining, is important and has led to the defining of two characterizing functions to distinguish these cases (Shapley & Shubik, unpublished, Chap. 3).

The availability of a directed threat is closely related to possibilities of blackmail and pressures to break up coalitions by

⁴ Note that the value 1 is the worth of being fourth and the value of 4 is the worth of being first

concentrating on one part of the coalition. Arguments against integration often show this feature where a lower middle-class white might argue against a rich, liberal white by saying, "You can afford to favor integration because it will not affect either your school or district but will affect mine."

Two- and three-person games of status

A two-person game of status is a direct duel whose outcome is determined by the relative sizes of the winnings obtained from the difference game. It is an inessential game in the sense that talk or bargaining or the formation of coalitions are not relevant to the outcome.

There are only four different $(0, 1)$ normalized three-person games of status. They arise from four games played for points which can be described by the following characteristic functions, where the "1" means that the coalition can attain both first and second place:

$$v(1) = v(2) = v(3) = 0 \ 0 \ 0 \ 0$$

$$v(1, 2) = 0 \ 0 \ 0 \ 1$$

$$v(1, 3) = 0 \ 0 \ 1 \ 1$$

$$v(2, 3) = 0 \ 1 \ 1 \ 1$$

$$v(1, 2, 3) = 1 \ 1 \ 1 \ 1.$$

The first three games all have veto players. In the first game every player has a veto, hence only the three-person coalition is able to obtain anything. In the second game, Players 2 and 3 have vetoes, and in the third, only Player 3 has the veto. A player has a veto if his absence from a coalition causes that coalition to obtain nothing. The first game is also known as the deterrence game because unless everyone agrees, they all obtain nothing. This is as though each participant had a weapon that would destroy everything unless an agreement were reached.

The fourth game has no veto player and any two players can obtain as much as all three. This has the same structure as the simple majority voting game. From the viewpoint of status, this game is directly equivalent to any one of the class of three-person games described by

$$v(1) = v(2) = v(3) = 0$$

$$v(1, 2) = a, \quad v(1, 3) = b, \quad v(2, 3) = c$$

$$v(1, 2, 3) = 1$$

where a , b , and c are all greater than zero. In terms of the game of status this means that any two players can obtain first and second places; thus, although the point score game is nonsymmetric, the game of status is symmetric and has the following characterizing functions:

$$F(1) = \{1\}$$

$$F(2) = \{2, 3\} \text{ or } \{3, 2\}$$

$$F(3) = \{3, 2, 1\}, \text{ and all other permutations.}$$

WHAT IS MEANT BY A SOLUTION?

A solution to a game may be regarded either as a prediction of what is going to happen or as a prescription of what should happen if certain desiderata are to be met. There are many extremely different solution concepts which have been suggested for n -person games. Descriptions of most of them are to be found elsewhere (Shapley & Shubik, unpublished, Chaps. 1 & 3; Luce & Raiffa, 1957). The discussion in this paper is limited to a nontechnical commentary on three well developed solution concepts. They are: (1) the core; (2) the value, and (3) the stable set solutions (Shapley & Shubik, unpublished, Chaps. 1 and 3). The first stresses group rationality and power, the second is addressed more to equity and symmetry conditions, and the third, to social stability. The Appendix provides somewhat more technical definitions and examples of these three solutions.

Consider the deterrence game where unless all players cooperate, everyone obtains nothing. Any division of the gains from cooperation is consistent with group rationality, because no set of players acting by themselves can obtain a more favorable outcome. On the other hand, in the simple majority voting game, any two players can obtain as much as all three. There is absolutely no way in which the gains can be divided without violating the group rationality of some coalition in the sense that both members of that coalition could improve the

outcome to themselves by failing to cooperate with the third. For example, suppose the split $(\frac{1}{2}, \frac{1}{2}, 0)$ were suggested, giving the first two players $\frac{1}{2}$ each and the third player nothing. The third player could immediately suggest a deal to the second opting for the outcome of $(0, \frac{3}{4}, \frac{1}{4})$.

In the deterrence game, there is always a distribution of proceeds such that no subgroup of society by acting alone can obtain more. When this is the case, the game is said to possess a (nonempty) core. In the simple majority voting game, there is absolutely no way of dividing the proceeds which does not give at least one coalition less than it could obtain by acting independently. Such a game has an empty core. Most political and social problems appear to be best represented by games without cores.

The value awards each individual a fair share based upon what he contributes to any coalition he could enter. This fair share is calculated by averaging the contribution of the individual not only to every coalition, but also by calculating the different contributions he might make by entering a coalition at a different time. (For example, even though all doctors might be equally good, the contribution made by the first one to enter a community will be larger than the contribution made by the tenth one.)

The value calculation (see Appendix) applied to the characteristic functions of the four games for points described previously would give values respectively of: $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$; $(0, \frac{1}{2}, \frac{1}{2})$; $(\frac{1}{6}, \frac{1}{6}, \frac{2}{3})$ and $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The first and fourth games are completely symmetric and so is the value. In the second game, it is evident that the first player never contributes anything to any coalition and hence, he is awarded nothing. In the third game, the key player is the third, and he obtains almost all of the value. The value must be modified to apply to games of status, in particular, and games without side payments, in general; this can be done, however (Shapley & Shubik, 1969).

The stable set solution in general does not consist of a single outcome, but of a set of outcomes that satisfy conditions of stability such that no group can argue that it strictly prefers one outcome in the solution to an-

other and is in a position to be able to enforce it. Furthermore, for any outcome not in the solution, there is always one in the solution such that some group does prefer the one in the solution, and is in a position to do something about it. Except in the Appendix, we do not investigate the mathematical properties of stable sets further.

The concept of social stability is an elusive one, and even in the highly restricted context of game theoretic models, it is hard to characterize. Using the von Neumann and Morgenstern concept of social stability, Lucas (1968) has been able to demonstrate that not all games will have a set of socially stable outcomes. It is conjectured that all games of status will have a socially stable solution. A game of status may be regarded as a generalization of a simple game (Shapley, 1962), and it is known that all such games possess a stable set solution (Shapley, 1968).

Games of status and the core

Any game of status that does not have a veto player does not have a core. This follows immediately from looking at the 1, $n - 1$ and n -person coalitions. The characteristic function of the game for points behind any game of status without a veto player is of the form:

$$\begin{aligned} v(1) &= v(2) = \dots = v(n) = 0 \\ &\vdots \\ v(\overline{n-1}) &= 1 \\ v(\bar{n}) &= 1 \end{aligned}$$

where $\overline{n-1}$ and \bar{n} stand for a set of any $n - 1$ or n players ($v(n)$ is the value of the set consisting of Player n). This is a direct extension of the fourth game in the previous section and will not have a core regardless of the values assigned to coalitions of size smaller than $n - 1$ or greater than 1.

Other solutions and economic externalities

In the recent literature on oligopoly theory several new and highly competitive solution concepts have been suggested. All of them are based upon the concern of the individual with how much others are obtaining as well as the amount he obtains himself. It is

worth noting that this concern with the performance of others is not necessarily always a manifestation of greed or of concern, but is a rather natural way to start to construct a measure of performance in situations where objective measures are somewhat difficult to attain. In Wall Street, for example, much of industry analysis is based upon comparisons between firms. This stress very often leads to measures based on "How am I doing in comparison with members of my peer group?" It is evident that a high emphasis on relative measures can lead to destructively competitive behavior. Maximin the difference is an example of destructive behavior in a two-person game. Games of status are a generalization within a social context, i.e., explicit cooperation among groups is expected, although each individual is still striving for status.

In the context of bureaucracy and economics, three other generalizations (Levitan & Shubik, 1967) of a noncooperative or individualistic nature have been considered. (Noncooperative means that the individuals act without debate or the forming of coalitions.) The first is dedicated to those who grade on the curve. It is beat the average and is naturally defined when the individual units are more or less of the same size and with the same apparant endowments—such as individuals taking an examination, or firms that are of the same size. This can be expressed formally as:

$$\max \left(P_i - \frac{1}{n-1} \sum_{j \neq i} P_j \right).$$

This requires less information and no bargaining and hence, has a far less Byzantine flavor than a game of status, but is suitable for the individual in the mass society when striving for promotion or recognition.

The two other solution concepts are maximize market share and maximize profit share. These are more specially aimed at the study of oligopolistic behavior and are not considered further here.

Even the most unreconstructed, utilitarian economist will acknowledge that individual preferences are not necessarily independent. Thus, the fraternity will always pause for at least two minutes when expounding the

gospel in order to talk about Veblen effects such as conspicuous consumption. The inter-linking of one individual's welfare with that of another may be referred to in terms of external economies or diseconomies, which is another way of saying that what *A* does affects *B* or "no man is an island unto himself." Much of basic economic theory has been based upon constructive competition by inner-directed individuals where the inter-linkage through preferences is not present. For many problems of welfare this is not adequate.

OTHER CONSIDERATIONS AND CONCLUSIONS

The level of perception

One of the difficulties with the concept of a game of status as formulated is the extreme sensitivity of the solutions to small changes in points. A more reasonable approach would involve introducing thresholds for perceptible differences; thus, if *A* has half a point more than *B*, the odds are that they will be judged to be equal.

Games of status and welfare

It has been observed that the United States, more than any other country in the world, is a one number society. It would be nice to have a single index to express the friendliness of nations or the power of groups. In some societies a man's wealth may be used as the magic number to sum him up. For many purposes, however, a single number is inadequate and the concept of *A* being better or kinder or more powerful than *B* can only be made meaningful in several dimensions of comparison. For example, a society might have four castes: priests, warriors, traders, and peasants, ranked in that order. It might be possible to have a trader who is richer than a warrior, yet the warrior still has higher status. In the long run, there might be a trade-off so that money and a marriageable daughter bring status eventually. However, in the short run, the immediate trade-off between money and status may be small, as many a nouveau riche has found out.

It is technically possible to formulate a more generalized form of game of status in

which several different types of points are played for initially, and in which there is imperfect conversion among them. Such a game would be hard to analyze and is not formulated at this time. Nevertheless, attempts to formalize models of this type should be worthwhile eventually, even if they serve only to provide a methodological device to help specify the dimensions needed to describe the situation. For example, it would be desirable to give complete operational meaning to statements such as: "Given that his status is not threatened, his major concern is for the welfare of others."

So what?

There are many aspects of development, welfare, and the treatment of minority groups that are not best analyzed in economic terms only, even though the economic component of the problems at hand is large. Good theorizing is a way of coding information parsimoniously, and even when it turns out to be wrong, it may serve as an aid to focus on the relevant. The models offered here are merely part of a first essay to adapt or tailor game-theoretic reasoning to sociological problems and to suggest the possibility of formal models involving mixed socio-economic goals.

APPENDIX

In order to explain the various concepts of solution, we must also define what is meant by the following terms: an imputation, an effective set of players, and domination of one imputation by another.⁵

The characteristic function specifies the worth that a coalition can achieve if it limits its trades strictly to itself. Mathematically it is a function $v(S)$ defined on sets of players S , with the properties

$$v(\emptyset) = 0$$

$$v(S \cup T) \geq v(S) + v(T), \text{ whenever } S \cap T = \emptyset$$

The first condition merely states that the amount achievable by the null set is nothing. The second condition is the fundamental economic

⁵ Much of the following exposition on the characteristic function, core, and von Neumann and Morgenstern stable set solution comes from a paper by Shapley and Shubik. This material has been slightly rewritten and is presented in order to make this article somewhat more self-contained.

property of superadditivity: if two separate groups having commerce only among themselves are joined together, the resultant group is at least as effective as were the two independent groups. Beyond these two conditions, there is nothing more that can be said a priori about a characteristic function.

If we denote the set of all players in a game by N , then $v(N)$ specifies the total amount that the whole group can obtain by cooperation. A reasonable form of cooperative behavior would be for the players to agree to maximize jointly,⁶ and then to decide how the proceeds are to be apportioned, or imputed. We define an imputation α to be a division of the proceeds from the jointly optimal play of the game among all the n players

$$\alpha = (\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n),$$

where

$$\alpha_i \geq v(i) \text{ and } \sum_{i=1}^n \alpha_i = v(N).$$

The condition $\alpha_i \geq v(i)$ embodies the principle that no individual will ever consent to a division that yields him less than he could obtain by acting by himself. It is often convenient to normalize the individual scales so that $v(i) = 0$.

A set of players is said to be effective for an imputation if by themselves they can obtain at least as much as they are assigned in that imputation. Symbolically, S is effective for α if and only if

$$v(S) \geq \sum_{i \in S} \alpha_i.$$

If $>$ rather than $=$ holds, we shall say that S is strictly effective.

An imputation α dominates an imputation β if there exists an effective set S for α such that all members of S , $\alpha_i > \beta_i$. Following the notation of von Neumann and Morgenstern (1944), we write

$$\alpha \succ \beta.$$

In other words, if a set S of players is in a position to obtain by independent action the amounts that they are offered in the imputation α , and if, when they compare the amounts offered in α to the amounts offered in β , all of them prefer the former, then α dominates β . There is a potential coalition that prefers α to β and is in a position to do something about it. Note that S is necessarily strictly effective for β , the dominated imputation.

Core and stable set

Finally, we may define two solution concepts. The core of an n -person game is the set of un-

⁶ In this Appendix, except when otherwise stated, all solution concepts are described for the transferable utility case. This is only for simplicity. Equivalent but more unwieldy definitions can be given for the nontransferable case. Both the basic concepts and almost all of the results are essentially equivalent.

dominated imputations, if any. A von Neumann-Morgenstern solution or stable set on the other hand, consists of a set of imputations that do not dominate each other, but which collectively dominate all alternative imputations. There is at most one core, but there may be many solutions. All solutions contain the core, if it exists.

Some examples

A series of simple, three-person games will illustrate these concepts. Consider first, the game in which any player acting by himself obtains nothing, but any pair of players acting in concert can demand three units to share between them, while all three players in coalition are also awarded three. The characteristic function of this game is

$$\begin{aligned} v(\emptyset) &= 0, \\ v(\bar{1}) &= v(\bar{2}) = v(\bar{3}) = 0, \\ v(\bar{12}) &= v(\bar{13}) = v(\bar{23}) = 3, \\ v(\bar{123}) &= 3, \end{aligned}$$

where $\bar{12}$ means the set consisting of Players 1 and 2.

We may represent the imputations in this game by triangular coordinates, as shown in Fig. 1. The vertices P_1, P_2, P_3 represent the imputations $(3, 0, 0), (0, 3, 0),$ and $(0, 0, 3),$ respectively. The point $\omega = (1, 1, 1)$ is the center of the triangle. Consider the two imputations $\alpha = (1.9, 0, 1.1)$ and $\beta = (0, 1.5, 1.5)$. The set $\bar{23}$ is effective for $\beta,$ and furthermore, both 2 and 3 are better off in β than in $\alpha.$ Hence $\beta \xi \alpha.$

The trio of imputations $\beta, \gamma,$ and δ forms a solution set to this particular game. Any other imputation gives two of the players less than 1.5 apiece, and thus is dominated by one of these three imputations, but the three do not dominate each other. (There are other solution sets, which we need not discuss.) This game has no core, since the imputations $\beta, \gamma,$ and $\delta,$ dominating all the rest, are themselves dominated by others. For example, the im-

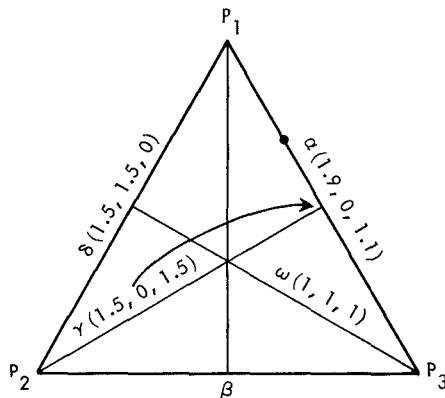


FIG. 1.

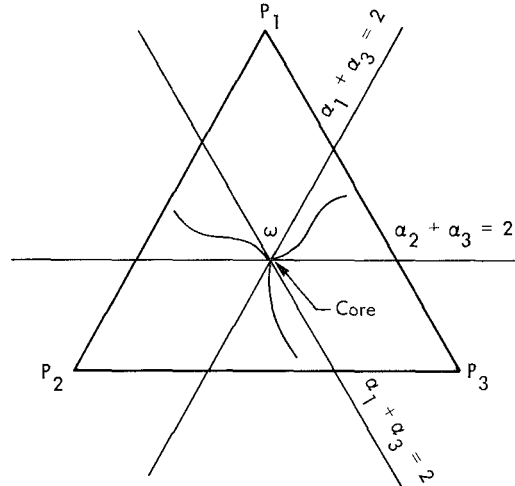


FIG. 2.

putation $\alpha,$ which was dominated by $\beta,$ in turn dominates δ via the effective set $\bar{13}.$ Note that domination is not a transitive relation $\beta \xi \alpha$ and $\alpha \xi \delta$ do not entail $\beta \xi \delta.$

We now consider three closely related games, differing from the previous one only in what the two-person coalitions obtain. In the first variant we have⁷

$$v(\bar{12}) = v(\bar{13}) = v(\bar{23}) = 0.$$

In this case, all imputations are in the core. The only set of players that is effective, for most imputations, is the three-person set, however, this is useless for domination, since on examining the distribution of welfare from the viewpoint of all three players, we see that if one player prefers one of two imputations, then at least one of the other players will prefer the other, the sum of the allotments being constant. In fact, it suffices to point out that no set of players is strictly effective for any imputation—hence, there is no domination. The core is, therefore, as large as possible, and is also the unique von Neumann-Morgenstern solution.

⁷ This all-or-nothing type of characteristic function, like the previous one, is associated more with political than with economic processes. The previous game was a majority-take-all situation; the present one is a veto situation, since, if one member wishes to be the dog in the manger, he can prevent the others from obtaining any payoff. In economics, such extremes—called simple games—are not typical. We shall presently consider variants in which the two-person coalitions obtain intermediate amounts, reflecting the more usual situation in which any new adherent to a coalition means added possibilities for profit.

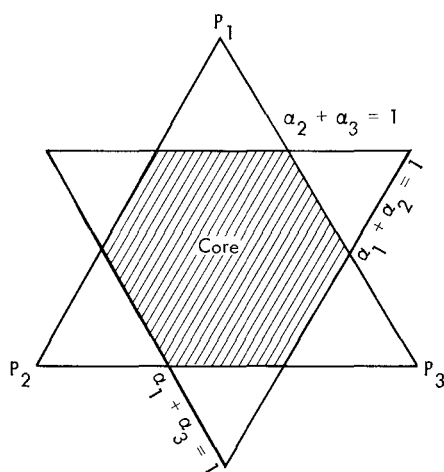


FIG. 3.

In our third example we assume

$$v(\bar{1}\bar{2}) = v(\bar{1}\bar{3}) = v(\bar{2}\bar{3}) = 2.$$

As shown in Fig 2, the lines that describe the amount obtainable by each coalition of two players intersect in a single point, the imputation ω with coordinates (1, 1, 1). This is the only undominated imputation of the game, and thus constitutes a single-point core. Since ω fails to dominate the three small triangles adjoining it in the diagram, however, it is not a von Neumann-Morgenstern solution by itself. To get a solution, we must add some more or less arbitrary curves, as shown, traversing the three triangular regions.

In the final variant, we assume that the two-person coalitions are only half as profitable as in the preceding example. That is, we have

$$v(\bar{1}\bar{2}) = v(\bar{1}\bar{3}) = v(\bar{2}\bar{3}) = 1.$$

The lines indicating the ranges of effectiveness of these coalitions are spread apart, as shown in Fig. 3, revealing a large, hexagonal core. All imputations in that area are undominated. As in the second example, this core is the unique solution.

A superficial examination of these four examples suggests a relationship between the size of the core and the fatness of the coalitions in a game, i.e., how much they can promise their members per capita as compared to the per capita amount available in the whole game. In all four instances, the latter amount was $v(\bar{1}\bar{2}\bar{3})/3 = 1$. Denote $v(\bar{i}\bar{j})/2$ by f_2 . In the first game, f_2 was 1.5, which is greater than 1, and there was no core. In the third game, f_2 was exactly 1, and the core was a single point. In the fourth game, f_2 was 1/2, and there was a large core, while in the second game f_2 was 0, and every imputation was in the core.

Of course, in a less symmetric situation, this principle would not reveal itself in such a clean-cut manner. However, a general rule of thumb

seems to persist: The more power there is in the hands of the middle-sized groups, the more narrowly circumscribed is the range of outcome of the cooperative game

When we do not permit the transfer of utility, it is no longer possible to regard the amount attainable by a coalition as a single number, yet cores and solutions can still be defined and a comparison of gains in various coalitions can still be made in a vectorial sense.

The value

Given the description of a game by means of its characteristic function, Shapley has suggested a method to impute a unique value to each player. One way of regarding this value is to consider that a priori all coalitions are equally likely. Furthermore, the probabilities of the order in which an individual joins a coalition are the same, e.g., in a three-person coalition, Player i has the same probability of being the first, second, or third to join.

We consider every possible order in which every individual can enter every coalition and we credit him with his incremental contribution to the coalition. In terms of the characteristic function coalition S and Player i this is

$$v(S) - v(S - \{i\})$$

Adding all his contributions together we average them and award the player that amount, ϕ_i , where:

$$\phi_i = \sum_{\text{all } S \subset N} \frac{(s-1)!(n-s)!}{n!} \cdot [v(S) - v(S - \{i\})]$$

The value may be arrived at through a series of axioms which reflect basic concepts of symmetry and fair division. Various criticisms have been made to the effect that the value may not be a desirable fair division scheme. For the most part, they are directed toward the difficulties inherent in the formulation of the characteristic function and the concept of threat. This will be referred to again after the examples.

We consider two voting games with four players with votes distributed 2, 1, 1 and 1. In the first game, there is no protection of the minority; any winning coalition can take all. In the second game, the minority is protected by a pro rata rule. The first game is known as a simple game—all values of the characteristic function are either 0 or 1.

Call the players a, b, c, d . We use the notation $\{a, b\}$ to represent the set of a and b , and for brevity $\bar{1}, \bar{2}$ or $\bar{3}$ and $\{a, \bar{1}\}$ etc., to represent a set of 1, 2 or 3 of the one-vote players and $\{a, \bar{1}\}$ the set consisting of a and one other. The characteristic function is:

$$\begin{aligned} v(\bar{1}) &= v(\{a\}) = 0 \\ v(\bar{2}) &= 0 \\ v(\{a, \bar{1}\}) &= 1 \end{aligned}$$

$$v(\bar{3}) = 1$$

$$v(\{a, \bar{2}\}) = 1$$

$$v(\{a, \bar{3}\}) = 1.$$

The characteristic function for the second game is

$$v(\bar{1}) = v(\{a\}) = 0$$

$$v(\bar{2}) = 0$$

$$v(\{a, \bar{1}\}) = 3/5$$

$$v(\bar{3}) = 3/5.$$

$$v(\{a, \bar{2}\}) = 4/5.$$

$$v(\{a, \bar{3}\}) = 1.$$

In the first game there is no core as can be seen by the inconsistency of requiring

$$\alpha_1 + \alpha_2 + \alpha_4 \geq 1$$

$$\alpha_1 + \alpha_2 + \alpha_3 \geq 1 \quad \alpha_1 + \alpha_2 \geq 1$$

$$\alpha_1 + \alpha_3 + \alpha_4 \geq 1 \quad \alpha_1 + \alpha_3 \geq 1$$

$$\alpha_2 + \alpha_3 + \alpha_4 \geq 1 \quad \alpha_1 + \alpha_4 \geq 1$$

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$$

where $\alpha_1, \alpha_2, \alpha_3$ and α_4 are the shares awarded to $a, b, c,$ and d respectively.

In the second game the conditions for the core are given by:

$$\alpha_2 + \alpha_3 + \alpha_4 \geq 3/5 \quad \alpha_1 + \alpha_2 \geq 3/5$$

$$\alpha_1 + \alpha_3 + \alpha_4 \geq 4/5 \quad \alpha_1 + \alpha_3 \geq 3/5$$

$$\alpha_1 + \alpha_2 + \alpha_3 \geq 4/5 \quad \alpha_1 + \alpha_4 \geq 3/5$$

$$\alpha_1 + \alpha_2 + \alpha_4 \geq 4/5$$

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1.$$

These are satisfied by the imputation ($\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}$).

In the first game, as an individual adds to a coalition only if he turns it from losing to winning, the following simple scheme illustrates and calculates the value.

1211; for Player a	6 × 1 = 6
1121; for Player a	6 × 1 = 6
1112; for Player a	6 × 0 = 0
2111; for Player a	6 × 0 = 0
	—
	12.

Total number of cases is 24.

$$\phi_a = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}, \quad \phi_b = \phi_c = \phi_d = \frac{1}{6}.$$

The imputation representing the value is ($\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}$).

In the scheme above, the dot indicates the pivotal player, i.e., the man who changes defeat to victory. Each line-up can occur six ways. For example, in the case 1211 we have $bacd, cabd, dacb, badc, cadb,$ and $dabc$. The player with two votes is pivotal in 12 out of 24 cases

In the second example we must also take into account the fact that a player joining an already winning coalition still makes a contribution.

1211	for Player a	6 × 3/5
1121	for Player a	6 × 4/5
1112	for Player a	6 × 2/5
2111	for Player a	6 × 0

$$54/5,$$

$$\phi_a = \frac{1}{24} \times \frac{54}{5} = \frac{9}{20}, \quad \phi_b = \phi_c = \phi_d = \frac{1}{60}$$

The imputation representing the value is ($\frac{9}{20}, \frac{1}{60}, \frac{1}{60}, \frac{1}{60}$)

The shift down in his value from that in the previous game reflects the effect of the protection of minority rights.

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